EXTREMAL POINTS, CRITICAL POINTS, AND SADDLE POINTS OF ANALYTIC FUNCTIONS

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Abstract

This paper is about the relation among critical points, saddle points and extremal points of complex analytic functions. We will show this relation is different than the one that we know in the case of real analytic functions and we will mainly deal with non-constant complex analytic functions. First we introduce some basic facts about these functions and discuss their differentiability conditions on open sets. Then, to include the boundary conditions and to be more specialized, we change our domain to compact sets. At the end of the paper, an interesting application about the estimation of certain real sums by using contour integral is exhibited. The application is interesting because the contour includes saddle points of the function. The value of the contour integral is estimated by using the *Maximum Likelihood (ML) estimate* technique which searches for the best bound to the value of the integral.

Öz

Bu makale karmaşık analitik fonksiyonların eyer noktaları, kritik noktaları ve bu fonksiyonların değerlerini en küçük yada en büyük yapan noktalar arasındaki ilişkiyi incelemektedir. Bu ilişkinin gerçel analitik fonksiyonlar için geçerli olandan farklı olduğunu göstereceğiz. Bunu yaparken genellikle karmaşık fonksiyonlarla çalışacağız. Öncelikle, bu fonksiyonlar hakkında bazı çıkarımlar sunacağız ve açık kümeler üzerinde türevlenebilirliklerini tartışacağız. Daha sonra, sınırlarla ilgili koşulları tartışmak için tıkız kümeleri tanım bölgesi olarak seçeceğiz. Makalenin sonunda gerçel toplamlara yol integrali kullanarak yaklaşmayı gösteren ilginç bir uygulama yer almaktadır. Bu uygulamanın ilginç olan yanı, yolun eyer noktaları içermesidir. Yol integralinin yaklaşık değeri, integrale en iyi üst sınırın bulunabildiği *maksimum likelihood kestirimi* tekniği kullanılarak hesaplanmıştır.

1. Introduction

This note explores the relation among critical points, saddle points and extremal points of complex analytic functions. We first give the definition of these specific points, and then mainly dealing with non-constant complex analytic functions, we introduce some important facts about the relation among them. After these steps, we conclude that this relation is different than the one that we know in the case of real analytic functions. At the end of the paper, an interesting application shows that certain real sums can be estimated by using contour integral which includes saddle points of the function.

2. Critical Points and Saddle Points of Analytic Functions

Using differentiation to find extrema is an important notion in the introductory calculus. For a real-valued function, the prime candidates for the points of its extrema are the zeros of its derivative. These candidates are called critical points. The same phenomenon also exists for the multivariable differentiable real-valued functions. The zeros of the gradient are labeled as critical points.

However, in general, these arguments are not valid for complex analytic functions. Since there is no natural total ordering of complex numbers, $|f|$ is used instead of f and we say that z_0 is a *local* (respectively, *absolute*) *extremal point* of a complex analytic function f if the real valued function $|f|$ attains a *local* (respectively, *absolute*) *extremum* at z_0. Also, we say that z_0 is a *saddle point* of f if it is a saddle point of $|f|$ (i.e., if a critical point but not an extremal point of $|f|$). As in the real case, the zeros of the derivative f' are called *critical points* of f. On the other hand, the critical points of an analytic function are more closely related to saddle points rather than to extremal points.

First of all, we introduce the *Maximum Modulus Principle* which says: for a holomorphic function f, the modulus $|f|$ cannot exhibit a local maximum which is properly within the domain of f. To be more formal:

Maximum Modulus Principle. Let $G \subset \mathbb{C}$ be a domain and $f : G \to \mathbb{C}$ be analytic. If there is any $a \in G$ with $|f(a)| \geq |f(z)|$, $\forall z \in G$, then f is constant.

As a result; the absolute value of a nonconstant analytic function on a domain $G \subset \mathbb{C}$ cannot have a local maximum point. So, if we let f be a nonconstant complex analytic function on an open set, *zeros* of f, if they exist, are the only local extremal points of f (where $|f|$ attains its absolute minimum). Thus, except for *zeros*, the absolute extrema of f on a compact set are always achieved on its boundary.

Before stating some facts about complex analytic functions, we should review a theorem that is analogous to the inverse function theorem for real numbers:

Theorem 1. Let f be analytic in a neighbourhood of a point z_0 and let $f(z_0) \neq 0$, then f has a local analytic inverse; there exists an analytic function g in a disc $|w - f(z_0)| < \rho$, $\rho > 0$ such that $f(g(w)) = w$ throughout this disc.

Now, let f be a nonconstant complex analytic function on an open set.

Fact 1. $|f|$ is not differentiable at z_0 whenever f has a *simple* zero at z_0 (i.e., when $f(z_0) = 0$ but $f'(z_0) \neq 0$).

Proof. Let $h(w) = |w|$, so we can write $|f| = h \circ f$. As $f' \neq 0$, f is one-to-one near z_0 and by Theorem 1, it has an analytic inverse, namely f^{-1}, defined on a neighbourhood $\mathcal{N}$ around $f(z_0)$. Hence, on $\mathcal{N}$, we can write $h = |f| \circ f^{-1}$. Since h is not differentiable at 0, $|f|$ is not differentiable at $f^{-1}(0) = z_0$.

Fact 2. z_0 is a critical point if and only if $|f|$ is differentiable at z_0 and has z_0 as a critical point.

Proof. First, let $f = u(x, y) + iv(x, y)$, where $u, v \in \mathbb{R}$ and let $g = |f|$. Assume g is differentiable at z_0 and $\nabla g(z_0) = \mathbf{0}$. We will try show z_0 is a critical point of f, i.e. $f'(z_0) = 0$. If $f(z_0) = 0$, then by our assumption about the differentiability of $g = |f|$, and by the result that we obtained from *Fact 1*, we conclude that $f'(z_0) = 0$. If $f(z_0) \neq 0$, we have:

$$g_x = \frac{(uu_x + vv_x)}{g}, \qquad g_y = \frac{(uu_y + vv_y)}{g}.$$

Since $\nabla g(z_0) = \mathbf{0}$, we have the linear system:

$$\begin{pmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0) \end{pmatrix} \begin{pmatrix} u(z_0) \\ v(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $f(z_0) \neq 0$, $u(z_0)$ and $v(z_0)$ are not both 0, the above equation implies:

$$det \begin{pmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0) \end{pmatrix} = 0.$$

So, we are left with $u_x(z_0)v_y(z_0) - u_y(z_0)v_x(z_0) = 0$. Since we know from Cauchy-Riemann equations that $u_x(z_0) = v_y(z_0)$ and $u_y(z_0) = -v_x(z_0)$ we get $u_x^2(z_0) + v_x^2(z_0) = 0$. It follows that $u_x(z_0)$ and $v_x(z_0)$ are both 0, because they are both real numbers. So, $f'(z_0) = u_x(z_0) + iv_x(z_0) = 0$.

For the converse; Assume $f'(z_0) = 0$. Since f is differentiable at z_0, it admits a power series expansion around z_0, for z sufficiently close to z_0, as follows:

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + \frac{f''(z_0)(z - z_0)^2}{2!} + R_2(z_0)$$

By our assumption and for some positive constant K, the above equation turns into:

$$f(z) - f(z_0) \leqslant K(z - z_0)^2,$$

By using triangle inequality, we get:

$$||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| \leq K|z - z_0|^2,$$

Dividing each side with $|z - z_0|$ and taking the *limit* as z tends to z_0 implies:

$$0 \leq \lim_{z \to z_0} \frac{||f(z)| - |f(z_0)||}{|z - z_0|} \leq \lim_{z \to z_0} K \left| z - z_0 \right|.$$

By *Squeeze law*, we conclude that $|f|$ is differentiable at z_0 and $\nabla |f| = \mathbf{0}$ there.

As a conclusion, the relationship between critical points and local extremal points for real valued functions does not hold for complex-valued analytic functions. The only local extremal points of a nonconstant complex analytic function f in a domain are its zeros and they need not to be critical points. A nonzero critical point of f, namely z_0, can be either a maximum or a minimum or a saddle point. However, z_0 cannot be a maximum because f attains its maximum on the boundary which is not included in the domain. Also, z_0 cannot be a minimum because zeros of f are the only minimum points of f. Hence, critical points are closely related with saddle points. Summarizing all these observations:

Proposition 1. A critical point of a nonconstant complex analytic function that is not a zero must be a saddle point. Conversely, a saddle point of an analytic function must be a critical point (and obviously not a zero).

3. Critical Points and Extremal Points On a Compact Set.

In this section, we will develop some of the previous ideas. While the zeros of a nonconstant analytic function f are the only local extremal points, namely local minimums, f can attain an absolute maximum on a compact set at a point on its boundary. Moreover, if f is non-vanishing, its absolute minimum is also assumed on the boundary. We will see that under some assumptions, if z_0 is such an absolute extremal point, then z_0 is not a critical point. Hence, being a critical point (and not a zero) of an analytic function is actually a guarantee of *not* being an *absolute* extremal point.

To investigate the behavior a function at a critical point on the boundary, we should now introduce a special kind of maximum principle which discusses the sign of the outward derivative of a point which is on the boundary of the domain where the function assumes its maximum:

Hopf Maximum Principle. Suppose $M = sup\{u(x) : x \in \Omega\} < \infty$.
(a) If $u(x_0) = M$ for some $x_0 \in \Omega$, then u is constant in Ω $[u(x) = M$ for all $x \in \Omega]$.
(b) If $u(x)$ is not constant in Ω, but $u(x_0) = M$ for some $x_0 \in \partial\Omega$ and $\partial u/\partial\nu$ exists at x_0, then $\dfrac{\partial u}{\partial \nu}(x_0) > 0$ holds, where ν is the unit exterior normal vector on $\partial\Omega$.

Proposition 2. Suppose that γ is a simple C^2-loop in $\mathbb{C}$ that D is the closure of the bounded component of $\mathbb{C} \setminus \gamma$, and that f is a nonconstant analytic function on an open set containing D. If $|f(z_0)|=\max_{z\in D}|f(z)|$, then $f'(z_0) \neq 0$, i.e. z_0 is not a critical point.

Proof. Define

$$F(z) = \frac{|f(z_0)|}{f(z_0)} f(z),$$

and let $F(z) = U(z) + iV(z)$ where $U, V \in \mathbb{R}$. We have $F(z_0) = |f(z_0)| \in \mathbb{R}$ so $F(z_0) = U(z_0)$. Also it is apparent that $|F(z)| = |f(z)|$. Hence,

$$U(z_0) = |f(z_0)| = \max_{z\in D}|f(z)| = \max_{z\in D}|F(z)|$$

From $U(z_0) = \max_{z\in D}|F(z)| = \max_{z\in D}|\sqrt{U^2(z) + V^2(z)}| \geq \max_{z\in D}|U(z)|$, we get $U(z_0) \geq \max_{z\in D}|U(z)|$, and since the converse inequality always holds, we conclude:

$$U(z_0) = \max_{z\in D}|U(z)|.$$

Now, U is a harmonic function because the definition of f and the relation between F and f implies U is a twice continuously differentiable function and it satisfies the Laplace equation which is easy to show by using Cauchy-Riemann equations:

Let $z = x + iy$, $x, y \in \mathbb{R}$. Since $U_x = V_y$, we have $U_{xx} = V_{yx}$ and since $U_y = -V_x$, we have $U_{yy} = V_{xy}$ which leads us to the result $U_{xx} + U_{yy} = 0$.

Since, U is a harmonic function and its maximum is on D is achieved at z_0 on $\partial D = \gamma$, the outward normal derivative $(\nabla U \cdot \mathbf{n})(z_0)$ is positive by *Hopf Maximum principle* (see[**2**, p.264]). [We can also conclude $\nabla U(z_0) \neq 0$ because it is a nonnegative multiple of the outward normal vector to γ at z_0. If it were not, $\nabla U(z_0)$ would have a nonzero component along the inward normal direction or along the tangential direction of γ which means U is greater than $U(z_0)$ some neighbourhood of z_0]. This result leads $U_x(z_0)$ and $U_y(z_0)$ are not both zero. Since $V_x = -U_y$,

$$F'(z_0) = U_x(z_0) + iV_x(z_0) \neq 0,$$

so $f'(z_0) \neq 0$ and we are done!

Alternative proof. Assume $f'(z_0) = 0$. For any $\xi \in \mathbb{C}$ and ξ of sufficiently small modulus we have

$$f(z_0 + \xi) = f(z_0) + \frac{f^{(k)}(z_0)}{k!}\xi^k + \cdots,$$

where k is the least integer with $f^{(k)}(z_0) \neq 0$ and the omitted terms are all of higher order in ξ than ξ^k. We compute:

$$\begin{aligned} |f(z_0+\xi)|^2 &= f(z_0+\xi)\overline{f(z_0+\xi)} \\ |f(z_0)|^2 &= |f(z_0)|^2 + f(z_0)\left(\overline{\frac{f^{(k)}(z_0)}{k!}\xi^k}\right) + \overline{f(z_0)}\frac{f^{(k)}(z_0)}{k!}\xi^k + \cdots \\ &= |f(z_0)|^2 + \frac{2}{k!}Re\left(\overline{f(z_0)}f^{(k)}(z_0)\xi^k\right) + \cdots . \end{aligned}$$

where we used the property: $a\bar{b} + \overline{a\bar{b}} = 2Re(a\bar{b})$ where $a, b \in \mathbb{C}$. By the way, we ignore the remaining terms of the equation above because the terms become much smaller as the order of the derivative of f becomes larger. We also have $f(z_0) \neq 0$ since $|f(z_0)| = \max_{z\in D}|f(z)|$.

Now, call $\overline{f(z_0)}f^{(k)}(z_0) = Ae^{i\alpha}$ with $A > 0$ and write $\xi = |\xi|e^{i\theta}$. Then,

$$\begin{aligned} |f(z_0+\xi)|^2 &= |f(z_0)|^2 + \frac{2}{k!}Re(Ae^{i\alpha}|\xi|^k e^{ik\theta}) + \cdots \\ &= |f(z_0)|^2 + \frac{2A}{k!}|\xi|^k \cos(k\theta + \alpha) + \cdots . \end{aligned}$$

Since $|f(z_0+\xi)|^2 - |f(z_0)|^2 = (|f(z_0+\xi)| - |f(z_0)|)(|f(z_0+\xi)| + |f(z_0)|)$, we conclude that $|f(z_0+\xi)| - |f(z_0)|$ has the same sign as $\cos(k\theta+\alpha)$ for ξ of sufficiently small modulus. Since $\cos\phi$ is defined to be *negative* for $\phi \in \left(\frac{(4j+1)\pi}{2}, \frac{(4j+3)\pi}{2}\right)$, we have $|f(z)| < |f(z_0)|$ *if z is in any of the k wedges of the form*:

$$\left\{z_0 + r_\theta e^{i\theta} : \theta \in \left(\frac{\pi + 4\pi j - 2\alpha}{2k}, \frac{3\pi + 4\pi j - 2\alpha}{2k}\right), r_\theta \in (0, \varepsilon_\theta)\right\} \qquad (1)$$

for some positive ε_θ and $j = 0, 1, \cdots, k-1$, and obviously $|f(z)| > |f(z_0)|$ *z is in the alternate wedges.* To complete the proof, first assume for the special case that D is a closed disc and z_0 lies on ∂D. So one of the k wedges defined in (1) *and* one of the alternate wedges must *both* intersect D (because z_0 is the value where $|f(z)|$ takes its maximum and *none* of the values of z in the k wedges can be z_0). Hence $|f|$ gets both larger and smaller values than $|f(z_0)|$ in D, so z_0 cannot be an extremal point of f on D. For the general case, let D be as in the Proposition 2, it is sufficient to observe the fact that, since ∂D is a C^2-curve, there exists a sufficiently small closed disc D_ε in D such that it intersects ∂D only at z_0 (see [**2**, p.264]). Similar to the disc argument, we conclude that $|f|$ assumes both smaller and larger values in D_ε; so z_0 cannot be an extremal point of f on D.

As a result of this proposition, we can conclude that if z_0 is a critical point of $|f|$ on D, then it cannot be an absolute maximum for the same function on the same region. This proposition shows how the critical points of an analytic function are more closely related to saddle points rather than to extremal points.

Remarks.

1. As the proposition states that f cannot achieve an absolute maximum value at a critical point, it is also true that 0 is the only value of $|f|$ at a critical point, which follows from the proof. This fact can be also derived by considering $1/f$ (which is analytic on an open set containing D where $f \neq 0$ on D).

2. The smoothness condition for ∂D guarantees the *interior disc condition* mentioned at the end of the proof. This result is also true whenever ∂D satisfies an *interior cone condition*: for all z_0 on ∂D there exists a cone in D of the form

$$\left\{ z_0 + re^{i\theta} : \theta \in [\alpha, \beta], r \in (0, \varepsilon) \right\}$$

   where $\varepsilon > 0$ and $\beta - \alpha > \pi/2$. This *interior cone condition* is valid because the wedges discussed in (1) has a maximum vertex angle of $\pi/2$ [since the vertex angle is π/k, where k is the least integer with $f^{(k)}(z_0) \neq 0$, and since $f'(z_0) = 0$, we conclude that $k = 2$]. Hence, the proposition would be valid if D were a polygon all of vertex angles were obtuse. On the other hand, without this *interior cone condition* there are some counterexamples to Proposition 2 as follows: in the unit square $\{z : Re(z), Im(z) \in [0, 1]\}$, $z^2 + i$ has *both* an absolute minimum *and* a critical point at 0, and $1/(z^2 + i)$ has *both* an absolute maximum *and* a critical point at 0.

3. From our results, the harmony of maximum points and critical points can be stated as follows:

Corollary. Suppose that f is a nonconstant analytic function on a simply connected open set R containing z_0 and that $f(z_0) \neq 0$. Then $f'(z_0) \neq 0$ *iff* there passes through z_0 a simple C^2-loop γ in R with $|f(z_0)| = \max_{z \in \gamma} |f(z)|$.

Proof. First of all, suppose that there exists such a C^2-loop γ and $|f(z_0)|$ is the absolute maximum value of $|f|$ one the compact set D which consists of γ and its *inside*. So, by Proposition 2, we conclude that $f'(z_0) \neq 0$.

For the converse, assume $f'(z_0) \neq 0$. By Theorem 1, f has an analytic inverse f^{-1} defined on a neighbourhood $\mathcal{N}$ around $f(z_0)$ and it is one-one near z_0. Hence we can let $\gamma = f^{-1}[\Gamma]$ where Γ is any simple C^2-loop in $\mathcal{N}$ through $f(z_0)$ such that $|f(z_0)| = \max_{\omega \in \Gamma} |\omega|$. (For instance, let Γ be a sufficiently small circle that has its center on the line segment between the origin and $f(z_0)$ and is tangent to the circle $\{\omega : |\omega| = |f(z_0)|\}$).

In addition to previous results, this corollary roughly states that a point z_0 where $|f|$ does not admit its maximum on a simply connected domain containing a twice differentiable curve, is a critical point of f.

4. Now, we will stress the necessity of the simple-connectivity condition in this corollary. Think of $f(z) = 1/(z^2 + 1)$ on the circle γ centered at i with radius 1. Clearly, $|f(0)| = \max_{z \in \gamma} |f(z)|$; but still $f'(0) = 0$. This phenomenon is because of the fact that f is not analytic on any simply connected domain containing γ. Another such example is demonstrated as an application.

4. An Application. Real-valued definite integrals and sums are associated with contour integrals in the complex plane. Most of the time, residue theorem is used for the evaluation of the value of a contour integral. However, when it is not possible, the contour integral can be evaluated by using the *Maximum Likelihood (ML) estimate*: $\left|\int_C f(z)dz\right| \leq \max_{z \in C} |f(z)| \cdot L(C)$, where $L(C)$ is the length of C.

An optimal upper bound for $\left|\int_C f(z)dz\right|$ is important in (ML) estimate technique because it gives rise to the search for the contour on which $|f|$ achieves a *minimal maximal value.* Furthermore, such a contour can include a saddle point of f. Before demonstrating such an argument, we should recall some basic facts about *Laurent series* which will be helpful for the calculation of the constant term of f in our example.

Definition. [Laurent Series.]

The Laurent series for a complex function $f(z)$ about a point c is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n$$

where the a_n are constants, defined by a line integral which is a generalization of Cauchy's integral formula:

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)dz}{(z - c)^{n+1}}.$$

Example.

As a roadmap; we will define a function $f = f(z)$, and expand it in terms of Laurent series. The constant term of f will become a link to the calculation of the minimal maximal value of f.

First observe that $\binom{n}{k}$ is the coefficient of z^k in the expansion of $(z+1)^n$ and $(-1)^k\binom{2n}{k}$ is the coefficient of z^{-k} in $(1-1/z)^{2n}$,

$$S_n = \sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{n}{k}$$

is the constant term in the expansion of $(z+1)^n(1-1/z)^{2n}$.

Let

$$f(z) = \frac{(z-1)^2(z+1)}{z^2}$$

So by using the second equality in the *Laurent Series* definition, S_n is equal to the contour integral

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{[f(z)]^n}{z}dz.$$

where r is any positive number (see[**3**, p.145]). To obtain an optimum estimate for $|S_n|$, we search for the minimum possible bound for $|f(z)|$ on any circle of radius $r > 0$.

Now, investigate the behaviour of $f(z)$ as z varies. Clearly, $f(z) \to \infty$ as $z \to \infty$ and as $z \to 0$. However, any circle of radius $r > 0$, f is continuous and $M(r) = \max_{|z|=r}|f(z)|$ exists. Moreover, $M(r)$ is clearly a continuous function of r, for $r > 0$. Since $M(r) \to \infty$ as $r \to \infty$ and $r \to 0$, there must be a particular value of R, with $M(R) = \min_{r>0} M(r)$. Our point is that the value of R, and (more importantly) the value of the point on $|z| = R$, call it z_0, with $|f(z_0)| = M(R)$ [$M(R)$ is also achieved at the point $\overline{z_0}$ on $|z| = R$].

We now examine $\nabla|f|$ at each of these points, z_0 and $\overline{z_0}$. The directional derivative of $|f|$ is zero there in the direction tangent to the circle $|z| = |z_0|$ because by the *Lagrange Multipliers* we have $\nabla|f|(z_0) = \lambda\nabla(|z| - |z_0|)(z_0)$ which implies that $\vec{\nabla}|f|(z_0)$ is parallel to $\vec{\nabla}(|z| = |z_0|)(z_0)$ so the gradient vector is perpendicular to the normal of the circle $|z| = |z_0|$ hence it should be zero in the direction tangent to this particular circle.

Furthermore, the directional derivative of $|f|$ is also zero in the direction normal to the circle $|z| = |z_0|$; otherwise we could find a circle of radius R' with $M(R') < M(R)$. This is nontrivial, but it can be proven as follows. Suppose, for e.g., that the directional derivative is positive at z_0. Then it is also positive for all points z_1 with $|Arg(z_1) - Arg(z_0)| < \delta$, δ fixed. It follows that $|f|$ would have a smaller value at the complex number $(1-t)z_0$ and at all points of the form $(1-t)z_1$ for t sufficiently small (so that the directional derivative remains positive). Then, by switching to $R' = (1-t)R$, for sufficiently small t, we will have $M(R') < M(R)$ since $|f|$ is less than $M(R)$ at all points on R within δ of $Arg(z_0)$ (or of $Arg(\overline{z_0})$) and $|f|$ will be less than $M(R)$ at all the other points of R' since they are sufficiently close to the compact set:

$$K = \{z : |z| = R,\ |Arg(z) - Arg(z_0)| \geq \delta \; and \; |Arg(z) - Arg(\overline{z_0})| \geq \delta\},$$

on which $|f|$ has a maximum value strictly less than $M(R)$ which contradicts with the definition of $M(R)$. Hence, we conclude that the directional derivative of $|f|$ is zero in the direction normal to the circle $|z| = |z_0|$. [**4**]

Moreover, we have the fact that the directional derivative is zero for two different vectors at a saddle point because the directional derivative can be both positive and negative at a saddle point. By the intermediate value theorem, there are two directions where the directional derivative vanishes. So, we conclude that z_0 and $\overline{z_0}$ are saddle points, hence critical points, of f (Proposition 1). To find z_0 we simply solve the equation $f'(z) = 0$, which is equivalent to $(z-1)(z^2+z+2) = 0$. Since 1 is a zero of f, the desired values of z_0 and $\overline{z_0}$ are $(-1 \pm \sqrt{7}i)/2$. Hence, $M = |f(z_0)| = |f(\overline{z_0})| = 2\sqrt{2}$, implying

$$S_n = \left| \sum_{k=0}^{n} (-1)^k \binom{2n}{k} \binom{n}{k} \right| \leq \left(2\sqrt{2}\right)^n.$$

As a conclusion, to apply (ML) estimate method to the real sum S_n, we first defined a function f and searched for a contour on which f attains its minimal maximum. Then we discussed the directional derivatives of these points where the function f gets its minimal maximum value and concluded that these points are saddle points of f.

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